AGE-DEPENDENT EQUATIONS WITH NON-LINEAR DIFFUSION

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ABSTRACT. We consider the well-posedness of models involving age structure and non-linear diffusion. Such problems arise in the study of population dynamics. It is shown how diffusion and age boundary conditions can be treated that depend non-linearly and possibly non-locally on the density itself. The abstract approach is depicted with examples.

1. INTRODUCTION

We consider abstract non-linear problems that naturally arise in the study of the dynamics of populations structured by age and spatial position (e.g. see [19] and the references therein). More precisely, we are interested in Banach-space-valued solutions to equations of the form

$$\partial_t u + \partial_a u = -A[\bar{u}](t)\, u - m\big(t,a,\bar{u}(t)\big)\, u\,, \qquad t>0\,, \quad a>0\,, \tag{1.1}$$

$$u(t,0) = B[u](t)\,, \qquad t>0\,, \tag{1.2}$$

$$u(0,a) = u^0(a)\,, \qquad a>0\,, \tag{1.3}$$

$$\bar{u}(t) = \int_0^\infty u(t,a)\, h(a)\, \mathrm{d}a\,, \qquad t>0\,. \tag{1.4}$$

The function $u=u(t,a)$ usually represents the population density of a certain specie at time $t>0$ and age $a>0$, so that $\bar{u}(t)$ in equation (1.4) is the (weighted) total population independent of age. The operator $A[\bar{u}](t)$ in equation (1.1) acts for a fixed function $\bar{u}$ and time t as a linear (and unbounded) operator on a Banach space E_0. In concrete applications, $A[\bar{u}](t)$ plays the role of non-linear diffusion. Equation (1.2) reflects the age-boundary conditions depending on the biological context.

The main features of equations (1.1)-(1.4) are the non-linear dependence of the operators A and B on the (total) density u. While a great part of the research so far focused on linear diffusion, it is the aim of this paper to present an approach in an abstract setting giving a framework for a larger class of problems of the form (1.1)-(1.4). This will not only provide us with some flexibility in choosing the underlying functional spaces in concrete applications, but also allows us to consider non-linear diffusion and age-boundary conditions that may depend locally or possibly non-locally with respect to time on the density u. The approach applies to general second order time-dependent elliptic operators on a smooth domain $\Omega\subset\mathbb{R}^n$, e.g. to operators of the form

$$A[\bar{u}](t)w = -\nabla_x\cdot\Big(D\big(\Phi(\bar{u})(t)\big)\nabla_x w\Big)$$

for some smooth function D with $D(z)\ge d_0>0$, $z\in\mathbb{R}$, subject to suitable boundary conditions on $\partial\Omega$. Here, the function Φ is a suitable function merely depending on $\bar{u}([0,t])$, in particular, $\Phi(\bar{u})(t)=\bar{u}(t)$ is possible. A reasonable choice is then $E_0=L_p(\Omega)$ with $p\in[1,\infty)$. As for the non-linear age-boundary condition (1.2), the operator B may also depend locally or non-locally on the density u. For instance, we may incorporate birth boundary conditions of the form

$$B[u](t) = \int_0^\infty b\big(t,a,\bar{u}(t)\big)\, u(t,a)\, \mathrm{d}a$$

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with some suitable birth modulus b (e.g., see [19]), or also age boundary conditions with history-dependent birth function of the form

$$B[u](t)=\int_0^\infty b\left(t,a,\int_{-\tau}^0 \bar{u}(t+\sigma)\mathrm{d}\sigma\right)\,u(t,a)\,\mathrm{d}a$$

as contemplated in [7], where $\tau>0$ is the maximal delay. We refer to our examples in Section 5.

In the next section, Section 2, we first list our assumptions and introduce the notion of a (generalized) solution to (1.1)-(1.4) before stating our main results on the well-posedness of (1.1)-(1.4). This section is then supplemented with further properties of the solution such as regularity, positivity, and global existence. The proof of the main result, Theorem 2.2, will be performed in Section 3, while the proofs of the additional properties will be given in Section 4. Finally, in Section 5 we briefly indicate how to apply these results in problems occurring in different situations of population dynamics.

We shall point out that other notions of solutions and other solution methods for age structured equations with linear diffusion were also introduced in literature, e.g. using integrated semigroups (see [11, 12] and the references therein) or using perturbation arguments (see [13, 14, 15]). For a similar approach as in the present paper we refer to [8, 9, 17, 18, 19]. We also refer to [4, 5, 6, 10] for other approaches to age structured equations with non-linear diffusion.

2. Main Results

In the following, we assume that E_1 and E_0 are Banach spaces such that E_1 is densely and continuously embedded in E_0. Furthermore, $(\cdot,\cdot)_\theta$ is for each $\theta\in(0,1)$ an admissible interpolation functor, that is, E_1 is densely embedded in each $E_\theta:=(E_0,E_1)_\theta$. Let $\mathcal{L}(E_1,E_0)$ denote the space of all bounded and linear operators from E_1 into E_0 equipped with the usual uniform operator norm. Given $\omega>0$ and $\kappa\geq 1$ we write

$$A\in\mathcal{H}(E_1,E_0;\kappa,\omega)$$

provided $A\in\mathcal{L}(E_1,E_0)$ is such that $\omega+A$ is an isomorphism from E_1 onto E_0 and satisfies

$$\frac{1}{\kappa}\;\leq\;\frac{\|(\lambda+A)u\|_{\mathcal{L}(E_1,E_0)}}{|\lambda|\,\|u\|_{E_0}+\|u\|_{E_1}}\;\leq\;\kappa\,,\quad Re\,\lambda\geq\omega\,,\quad u\in E_1\setminus\{0\}\,.$$

We set

$$\mathcal{H}(E_1,E_0):=\bigcup_{\substack{\kappa\geq 1\\ \omega>0}}\mathcal{H}(E_1,E_0;\kappa,\omega)\,,$$

which (equipped with the topology induced by the uniform operator norm) is an open subset of $\mathcal{L}(E_1,E_0)$. It is well known that $A\in\mathcal{H}(E_1,E_0)$ if and only if $-A$, considered as a linear operator in E_0 with domain E_1, is the generator of a strongly continuous analytic semigroup on E_0, e.g. see [1].

Next, we fix a function $g\in L^+_{\infty,loc}(\mathbb{R}^+)$ satisfying

$$0\;<\;g_0\;\leq\;g(a+b)\;\leq\;g_1\,g(a)\,g(b)\,,\quad a,b>0\,,\tag{2.1}$$

for some numbers $g_j>0$, and we introduce the Banach space

$$\mathbb{E}_\theta:=L_1\big(\mathbb{R}^+,E_\theta,g(a)\mathrm{d}a\big)\,.$$

If $T>0$, we put $I_T:=[0,T]$. Given a function $u\in\mathbb{E}_0^{I_T}$, we simply write $u(t,a)$ for $t\in I_T$ and $a>0$ instead of $u(t)(a)$. For an interval J we set $\dot{J}:=J\setminus\{0\}$.

For $\sigma\in\mathbb{R}$ and $\gamma\in[0,1]$ let $C_\sigma((0,T],\mathbb{E}_\gamma)$ be the space of all continuous functions $v:(0,T]\to\mathbb{E}_\gamma$ such that $t\mapsto t^\sigma v(t)$ stays bounded in the norm of $\mathbb{E}_\gamma$.

Throughout we suppose that there exists a number $\alpha\in[0,1)$ such that the following assumptions hold:

(A_1) The function $h \in L^+_{\infty,loc}(\mathbb{R}^+)$ satisfies $\overline{\lim}_{a\to\infty} \frac{h(a)}{g(a)} < \infty$, and there exists $\zeta > 0$ such that for each $T > 0$ there is $c(T) > 0$ with

$$|h(t+a) - h(t_*+a)| \le c(T)\, g(a)\, |t-t_*|^\zeta\ , \quad 0 \le t, t_* \le T\ , \quad a \ge 0\ .$$

(A_2) Given $T_0, R > 0$ and $\theta \in (0,1)$ there are numbers $\rho \in (0,1)$, $\omega > 0$, $\kappa \ge 1$, $\sigma \in \mathbb{R}$, and $c_0 > 0$ (depending possibly on θ, T_0, and R) such that for each $T \in (0, T_0]$ the operator $A = \big[\bar u \mapsto A[\bar u]\big]$ maps $C^\theta(I_T, E_\alpha)$ into $C^\rho(I_T, \mathcal{L}(E_1, E_0))$ and satisfies

$$\sigma + A[\bar u] \in C\big(I_T, \mathcal{H}(E_1, E_0; \kappa, \omega)\big)\ , \qquad \|A[\bar u]\|_{C^\rho(I_T,\mathcal{L}(E_1,E_0))} \le c_0\ , \tag{2.2}$$

and

$$\|A[\bar u] - A[\bar u_*]\|_{C(I_T,\mathcal{L}(E_1,E_0))} \le c_0\, \|\bar u - \bar u_*\|_{C(I_T,E_\alpha)} \tag{2.3}$$

for all $\bar u, \bar u_* \in C^\theta(I_T, E_\alpha)$ with $\|\bar u\|_{C^\theta(I_T,E_\alpha)} \le R$ and $\|\bar u_*\|_{C^\theta(I_T,E_\alpha)} \le R$. Moreover, if $0 < T < S$ and $\bar u, \bar u_* \in C(I_S, E_\alpha)$ with $u\big|_{I_T} = u_*\big|_{I_T}$, then $A[\bar u]\big|_{I_T} = A[\bar u_*]\big|_{I_T}$.

(A_3) There exists $\mu > 0$ such that, for $0 < T \le T_0$ and $R > 0$, the function B maps $C(I_T, \mathbb{E}_\alpha)$ into $C(I_T, E_\mu)$, and there exists some $c_0 = c_0(T_0, R) > 0$ such that

$$\|B[u] - B[u_*]\|_{C(I_T,E_\mu)} \le c_0\, \|u - u_*\|_{C(I_T,\mathbb{E}_\alpha)} \tag{2.4}$$

provided that $u, u_* \in C(I_T, \mathbb{E}_\alpha)$ with $\|u\|_{C(I_T,\mathbb{E}_\alpha)} \le R$ and $\|u_*\|_{C(I_T,\mathbb{E}_\alpha)} \le R$. In addition, if $0 < T < S$ and $u, u_* \in C(I_S, \mathbb{E}_\alpha)$ with $u\big|_{I_T} = u_*\big|_{I_T}$, then $B[u]\big|_{I_T} = B[u_*]\big|_{I_T}$.

(A_4) The function $m \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times E_\alpha, \mathbb{R})$ is such that, given $T > 0$ and $R > 0$, there exists $c_0 = c_0(T, R) > 0$ with

$$|m(t,a,\bar u) - m(t,a,\bar u_*)| \le c_0\, \|\bar u - \bar u_*\|_{E_\alpha}$$

and

$$e^{-m(t,a,\bar u)} \le c_0 \tag{2.5}$$

for $t \in I_T$, $a > 0$, and $\|\bar u\|_{E_\alpha}, \|\bar u_*\|_{E_\alpha} \le R$.

The latter assumptions in (A_2) and (A_3) guarantee that equations (1.1)-(1.4) pose a proper time evolution problem, that is, the solution depends at each time t only on the past but not on the future. In Section 5 we will give concrete examples for operators A and B satisfying (A_2) and (A_3), respectively. In particular, it will be shown that if A depends locally with respect to time on $\bar u$ and if E_1 is compactly embedded in E_0, then (A_2) is rather easy to verify in applications (see Proposition 5.1 and Corollary 5.2). Introducing the function g in the definition of the spaces $\mathbb{E}_\theta$ allows to give a meaning to (1.4) for $u \in \mathbb{E}_0^{I_T}$ in view of assumption (A_1). Also note that (2.5) is trivially satisfied if m is non-negative or bounded.

In order to introduce the notion of a solution to (1.1)-(1.4), we first observe that if $\bar u : I_T \to E_\alpha$ is Hölder continuous, then [1, II.Cor.4.4.2] and (2.2) ensure that $-A[\bar u]$ generates a unique evolution system $U_{A[\bar u]}(t,s)$, $0 \le s \le t \le T$, on E_0.

Definition 2.1. *A function $u \in C(J, \mathbb{E}_\alpha)$ is a generalized solution to* (1.1)-(1.4) *on an interval J provided that*

(i) *$\bar u : J \to E_\alpha$ is Hölder continuous,*

(ii) *u satisfies*

$$u(t,a) = \begin{cases} e^{-\int_0^a m_{\bar u}(s+t-a,s)\,\mathrm{d}s}\, U_{A[\bar u]}(t,t-a)\, B[u](t-a)\ , & 0 \le a < t\ , \\ e^{-\int_0^t m_{\bar u}(s,s+a-t)\,\mathrm{d}s}\, U_{A[\bar u]}(t,0)\, u^0(a-t)\ , & 0 \le t < a\ , \end{cases}$$

for $t \in J$ and $a > 0$, where $m_{\bar u}(t,a) := m\big(t,a,\bar u(t)\big)$.

The notion of a generalized solution is derived by integrating (1.1)-(1.4) formally along characteristics. Proposition 2.5 below gives more details regarding further regularity of generalized solutions.

We first state an existence and uniqueness result for generalized solutions to (1.1)-(1.4).

Theorem 2.2. *Suppose $(A_1)-(A_4)$ with $\alpha\in[0,1)$ and let $0\le\alpha<\beta\le 1$. Then, given $u^0\in\mathbb{E}_\beta$, there exists a unique maximal generalized solution $u:=u(\cdot;u^0)$ to* (1.1)-(1.4) *on an interval $J:=J(u^0)$ with*

$$u\in C(J,\mathbb{E}_\beta)\cap C_{\upsilon-\beta}((0,T],\mathbb{E}_\upsilon)\ ,\quad \beta\le\upsilon\le 1\ ,\quad T\in\dot{J}\ .$$

Moreover,

$$\int_0^\infty u(\cdot,a)\,\tilde{h}(a)\,\mathrm{d}a\in C^{\zeta\wedge(\beta-\gamma)}(J,E_\gamma)$$

for $\gamma\in[\alpha,\beta)$ and any function $\tilde{h}$ satisfying (A_1). In addition,

$$\int_0^\infty u(\cdot,a)\,\mathrm{d}a\in C^1(\dot{J},E_0)\cap C(\dot{J},E_1)\ . \tag{2.6}$$

The maximal interval of existence, J, is open in $\mathbb{R}^+$, and if

$$\sup_{t\in J\cap[0,T]}\|u(t,\cdot)\|_{\mathbb{E}_\beta}<\infty\ ,\quad T>0\ , \tag{2.7}$$

then the solution exists globally, that is, $J=\mathbb{R}^+$.

A proof of this theorem will be given in Section 3. Before providing more properties of the generalized solution, we shall emphasize that the regularity assumptions on the operators A and B in (A_2) and (A_3) are imposed to overcome the difficulties induced by the quasi-linear structure of $A=A[\bar{u}]$. Indeed, in the case of "linear diffusion", that is, if $A=A(t)$ depends possibly on time but is independent of $\bar{u}$, less assumptions are required. For simplicity, we state the following remark for a function $m=m(t,a)$ that is independent of $\bar{u}$.

Remark 2.3. *Suppose that $A\in C^\rho(\mathbb{R}^+,\mathcal{H}(E_1,E_0))$ for some $\rho>0$, and for each $T>0$ let there be numbers $0\le\alpha\le\beta\le 1$ with $(\alpha,\beta)\neq(0,1)$ such that the function $B:C(I_T,\mathbb{E}_\beta)\to C(I_T,E_\alpha)$ is uniformly Lipschitz continuous on bounded sets and satisfies $B[u]|_{I_T}=B[u_*]|_{I_T}$ for $0<T<S$, $u,u_*\in C(I_S,\mathbb{E}_\beta)$, and $u|_{I_T}=u_*|_{I_T}$. If $m\in C(\mathbb{R}^+\times\mathbb{R}^+)$ is bounded, then the problem*

$$\begin{aligned}\partial_t u+\partial_a u&=-A(t)\,u-m(t,a)\,u\ ,\quad t>0\ ,\quad a>0\ ,\\ u(t,0)&=B[u](t)\ ,\quad t>0\ ,\\ u(0,a)&=u^0(a)\ ,\quad a>0\ ,\end{aligned}$$

admits for each $u^0\in\mathbb{E}_\beta$ a unique maximal generalized solution $u\in C(J,\mathbb{E}_\beta)$, which exists globally if (2.7) *holds.*

A proof of this remark follows along the lines of the proof of Theorem 2.2 and we thus omit details.

We now give additional properties of the generalized solution. For the rest of this section, we suppose the assumptions of Theorem 2.2, and we fix $u^0\in\mathbb{E}_\beta$ and let $u=u(\cdot;u^0)\in C(J,\mathbb{E}_\beta)\cap C_{\upsilon-\beta}((0,T],\mathbb{E}_\upsilon)$ for $T\in\dot{J}$ and $\beta\le\upsilon\le 1$ denote the unique maximal generalized solution to (1.1)-(1.4) on $J=J(u^0)$ corresponding to u^0.

First we mention that the solution depends continuously on the initial value $u^0\in\mathbb{E}_\beta$. More precisely, we have

Corollary 2.4. *Given $u^0\in\mathbb{E}_\beta$ there exists $\delta>0$ and $T=T(u^0)>0$ such that $J(u_*^0)\supset[0,T]$ for every $u_*^0\in\mathbb{E}_\beta$ with $\left\|u^0-u_*^0\right\|_{\mathbb{E}_\beta}\le\delta$. Moreover, $u(\cdot;u_*^0)\to u(\cdot;u^0)$ in $C([0,T],\mathbb{E}_\beta)$ as $u_*^0\to u^0$ in $\mathbb{E}_\beta$.*

Next, we note that the solution possesses more regularity if the date are more regular.

Proposition 2.5. *Suppose that*

$$u^0 \in \mathbb{E}_\beta \cap C^1(\mathbb{R}^+, E_0) \cap C(\mathbb{R}^+, E_1) \, . \tag{2.8}$$

In addition, let

$$B[u] \in C^1(J, E_0) \cap C(J, E_1) \qquad \textit{and} \qquad m_{\bar{u}} \in C^{0,1}(J \times \mathbb{R}^+) \cup C^{1,0}(J \times \mathbb{R}^+) \, . \tag{2.9}$$

Then, for all $t \in \dot{J}$ and $a > 0$, we have

$$\partial_t u(t, \cdot) \, , \, \partial_a u(t, \cdot) \in C([0, t], E_0) \cap C((t, \infty), E_0) \, , \tag{2.10}$$

$$\partial_t u(\cdot, a) \, , \, \partial_a u(\cdot, a) \in C([0, a) \cap J, E_0) \cap C([a, \infty) \cap J, E_0) \, , \tag{2.11}$$

and u solves (1.1)-(1.3) *pointwise in E_0 for $t \neq a$.*

Since u represents a density in applications, one expects it to be non-negative. The next result establishes this positivity result if E_0 is an ordered B-space with positive cone E_0^+. In this case we put

$$\mathbb{E}_\theta^+ := L_1(\mathbb{R}^+, E_\theta^+, g(a)\mathrm{d}a) \quad \text{with} \quad E_\theta^+ := E_\theta \cap E_0^+ \, .$$

We refer to [1] for more information about operators on ordered B-spaces.

Proposition 2.6. *Suppose that E_0 is an ordered B-space with positive cone E_0^+. Given $T > 0$, $\theta > 0$, and $\bar{v} \in C^\theta([0, T], E_\alpha)$ let the linear operator $A[\bar{v}](t)$ be resolvent positive for each $t \in [0, T]$. Further suppose that B maps $C(I_T, \mathbb{E}_\alpha^+)$ into $C(I_T, E_\mu^+)$. Then $u^0 \in \mathbb{E}_\beta^+$ implies $u(t) \in \mathbb{E}_\beta^+$ for $t \in J$.*

We next focus on global existence. Due to the quasi-linear structure of equation (1.1) it is clearly not obvious how to derive estimates like (2.7) in general. The next result aims at providing conditions ensuring (2.7).

Proposition 2.7. *Let $\vartheta \in [0, 1]$ with $(\vartheta, \beta) \neq (0, 1)$. Suppose that for each $T > 0$ there are numbers $\varrho > 0$, $\sigma \in \mathbb{R}$, $\kappa \geq 1$, $\omega > 0$, and $c_1 > 0$ depending possibly on T such that*

$$\sigma + A[\bar{u}] \in C\big(J_T, \mathcal{H}(E_1, E_0; \kappa, \omega)\big) \tag{2.12}$$

with

$$\|A[\bar{u}](t) - A[\bar{u}](t_*)\|_{\mathcal{L}(E_1, E_0)} \leq c_1 |t - t_*|^\varrho \, , \quad t, t_* \in J_T \, , \tag{2.13}$$

and

$$\|B[u](t)\|_{E_\vartheta} \leq c_1 \big(1 + \max_{0 \leq \tau \leq t} \|u(\tau)\|_{\mathbb{E}_\beta}\big) \, , \quad t \in J_T \, , \tag{2.14}$$

where $J_T := J \cap [0, T]$ for $T > 0$. Further suppose that m a non-negative or bounded. Then the solution u exists globally, that is, $J = \mathbb{R}^+$.

Remark 2.8. *If the constant c_0 in* (2.4) *does not depend on R, then* (2.14) *is a consequence of* (2.4)*. Also, condition* (2.14) *may be replaced by*

$$\|B[u](t)\|_{E_\vartheta} \leq c_2 \big(1 + \|u(t)\|_{\mathbb{E}_\upsilon}\big) \, , \quad t \in J_T \, , \tag{2.15}$$

for $(0, 1) \neq (\vartheta, \upsilon) \in [0, 1]^2$ and some $c_2 = c_2(T) > 0$. This latter condition is slightly weaker than (2.14) *with respect to regularity since we may allow for $\upsilon > \beta$, but it somehow assumes B to depend locally on u with respect to time.*

For the proofs of Corollary 2.4, Propositions 2.5-2.7, and Remark 2.8 we refer to Section 4.

3. PROOF OF THEOREM 2.2

Given the assumptions of Theorem 2.2, let $\gamma\in[\alpha,\beta)$ be arbitrary and choose $\theta\in(0,\zeta\wedge(\beta-\gamma))$. We fix any $T_0>0$ and R with

$$R > \big(1+g_1\,\|g\|_{L_\infty(0,T_0)}\big)\ \|u^0\|_{\mathbb{E}_\beta}\max_{0\le s\le t\le T_0}\|U_{A[0]}(t,s)\|_{\mathcal{L}(E_\beta,E_\gamma)}\ . \tag{3.1}$$

For $T\in(0,T_0)$ set

$$\mathcal{V}_T := \big\{u\in C(I_T,\mathbb{E}_\gamma)\,;\ \|u(t)\|_{\mathbb{E}_\gamma}\le R+1\,,\ \|\bar u(t)-\bar u(t_*)\|_{E_\gamma}\le |t-t_*|^\theta\,,\ 0\le t,t_*\le T\big\}\ , \tag{3.2}$$

where

$$\bar v := \int_0^\infty v(a)\,h(a)\,\mathrm{d}a\ ,\quad v\in\mathbb{E}_0\ ,$$

and observe that $\mathcal{V}_T$, equipped with the topology induced by $C(I_T,\mathbb{E}_\gamma)$, is a complete metric space. Also note that (A_1) ensures the existence of a constant $c_1>0$ such that

$$0\le h(a)\le c_1 g(a)\ ,\quad a\ge 0\ , \tag{3.3}$$

and hence

$$\|\bar u\|_{E_\vartheta}\ \le\ c_1\,\|u\|_{\mathbb{E}_\vartheta}\ ,\quad u\in\mathbb{E}_\vartheta\ ,\quad \vartheta\in[0,1]\ . \tag{3.4}$$

In particular, due to the embedding $E_\gamma\hookrightarrow E_\alpha$ there is a constant $c(R)>0$ for which

$$\|\bar u\|_{C^\theta(I_T,E_\alpha)}\le c(R)\ ,\quad u\in\mathcal{V}_T\ ,$$

and thus there are numbers $\rho\in(0,1)$, $\omega>0$, $\kappa\ge 1$, $\sigma\in\mathbb{R}$, and $c_0>0$ depending on T_0 and R such that (2.2), (2.3) hold for $u,u_*\in\mathcal{V}_T$. Therefore, invoking Lemma II.5.1.3, Lemma II.5.1.4, and Equation (II.5.3.8) in [1], we conclude that there exists $c(T_0,R)>0$ such that unique evolution systems $U_{A[\bar u]}$ and $U_{A[\bar u_*]}$ on E_0 corresponding to any $u,u_*\in\mathcal{V}_T$ satisfy

$$\|U_{A[\bar u]}(t,s)\|_{\mathcal{L}(E_\sigma)}\ +\ (t-s)^{\tau-\sigma}\|U_{A[\bar u]}(t,s)\|_{\mathcal{L}(E_\upsilon,E_\tau)}\ \le\ c(T_0,R) \tag{3.5}$$

for $0\le s<t\le T$ and $0\le\sigma<\upsilon\le\tau\le 1$,

$$\|U_{A[\bar u]}(t,r)-U_{A[\bar u]}(s,r)\|_{\mathcal{L}(E_\tau,E_\upsilon)}\ \le\ c(T_0,R)\,(t-s)^{\tau-\upsilon} \tag{3.6}$$

for $0\le r<s<t\le T$ and $0<\upsilon\le\tau<1$, as well as

$$\begin{aligned}\|U_{A[\bar u]}(t,s)-U_{A[\bar u_*]}(t,s)\|_{\mathcal{L}(E_\sigma,E_\upsilon)}\ &\le\ c(T_0,R)\,(t-s)^{\sigma-\upsilon}\,\|\bar u-\bar u_*\|_{C(I_T,E_\alpha)}\\ &\le\ c(T_0,R)\,(t-s)^{\sigma-\upsilon}\,\|u-u_*\|_{\mathcal{V}_T}\end{aligned} \tag{3.7}$$

for $0\le s<t\le T$ and $0\le\sigma,\upsilon\le 1$ with $\sigma\ne 0$, $\upsilon\ne 1$.

Next, for $u,u_*\in\mathcal{V}_T$ we have $B[u]\in C(I_T,E_\mu)$ by (A_4) with

$$\begin{aligned}\|B[u]-B[u_*]\|_{C(I_T,E_\mu)}\ &\le\ c(T_0,R)\,\|u-u_*\|_{C(I_T,\mathbb{E}_\alpha)}\\ &\le\ c(T_0,R)\,\|u-u_*\|_{\mathcal{V}_T}\ ,\end{aligned} \tag{3.8}$$

whence

$$\|B[u]\|_{C(I_T,E_\mu)}\ \le\ c(T_0,R)\ . \tag{3.9}$$

Also note that, for $u,u_*\in\mathcal{V}_T$ and $t\in[0,T]$,

$$\begin{aligned}&\left|\int_0^a m_{\bar u}\ (s+t-a,s)\mathrm{d}s-\int_0^a m_{\bar u_*}(s+t-a,s)\mathrm{d}s\right|\\ &\qquad+\left|\int_0^t m_{\bar u}(s,s+b-t)\mathrm{d}s-\int_0^t m_{\bar u_*}(s,s+b-t)\mathrm{d}s\right|\ \le\ c(T_0,R)\,\|u-u_*\|_{\mathcal{V}_T}\end{aligned} \tag{3.10}$$

provided $0\le a\le t<b$. Defining Θ by

$$\Theta(u)(t,a)\ :=\ \begin{cases} e^{-\int_0^a m_{\bar u}(s+t-a,s)\mathrm{d}s}\,U_{A[\bar u]}(t,t-a)\,B[u](t-a)\ , & 0\le a<t\ ,\\ e^{-\int_0^t m_{\bar u}(s,s+a-t)\mathrm{d}s}\,U_{A[\bar u]}(t,0)\,u^0(a-t)\ , & 0\le t<a\ ,\end{cases}$$

for $0 \le t \le T$, $a > 0$, and $u \in \mathcal{V}_T$, we claim that $\Theta : \mathcal{V}_T \to \mathcal{V}_T$ is a contraction provided that $T = T(R) \in (0, T_0)$ is chosen sufficiently small. In the following, let $\bar{\mu} \in (0, \mu)$.

We first prove that $\Theta(u) \in C(I_T, \mathbb{E}_\beta) \hookrightarrow C(I_T, \mathbb{E}_\gamma)$ for $u \in \mathcal{V}_T$. To this end, observe that assumptions on g imply

$$g(a) \le g_1 \, \|g\|_{L_\infty(0,T_0)} \, g(a-t) \,, \quad a > t \,, \quad 0 \le t \le T \,. \tag{3.11}$$

Hence, recalling that $g \in L_{\infty,loc}(\mathbb{R}^+)$ and using (2.5), (3.5), (3.6), (3.9), and (3.11) we estimate for $u \in \mathcal{V}_T$ and $0 \le t \le t_* \le T$

$$\begin{aligned}
&\|\Theta(u)(t) - \Theta(u)(t_*)\|_{\mathbb{E}_\beta} \\
&\le \int_0^t \left| e^{-\int_0^a m_{\bar{u}}(s+t_*-a,s)\mathrm{d}s} - e^{-\int_0^a m_{\bar{u}}(s+t-a,s)\mathrm{d}s} \right| \left\| U_{A[\bar{u}]}(t_*, t_*-a) \right\|_{\mathcal{L}(E_\mu, E_\beta)} \\
&\qquad\qquad \times \|B[u](t_*-a)\|_{E_\mu} \, g(a) \, \mathrm{d}a \\
&\quad + \int_0^t \left| e^{-\int_0^a m_{\bar{u}}(s+t-a,s)\mathrm{d}s} \right| \left\| \left[U_{A[\bar{u}]}(t_*, t_*-a) - U_{A[\bar{u}]}(t, t-a) \right] B[u](t_*-a) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&\quad + \int_0^t \left| e^{-\int_0^a m_{\bar{u}}(s+t-a,s)\mathrm{d}s} \right| \left\| U_{A[\bar{u}]}(t, t-a) \right\|_{\mathcal{L}(E_\mu, E_\beta)} \|B[u](t_*-a) - B[u](t-a)\|_{E_\mu} \, g(a) \, \mathrm{d}a \\
&\quad + \int_t^{t_*} \left| e^{-\int_0^a m_{\bar{u}}(s+t_*-a,s)\mathrm{d}s} \right| \left\| U_{A[\bar{u}]}(t_*, t_*-a) \right\|_{\mathcal{L}(E_\mu, E_\beta)} \|B[u](t_*-a)\|_{E_\mu} \, g(a) \, \mathrm{d}a \\
&\quad + \int_t^{t^*} \left| e^{-\int_0^t m_{\bar{u}}(s,s+a-t)\mathrm{d}s} \right| \left\| U_{A[\bar{u}]}(t, 0) \right\|_{\mathcal{L}(E_\beta)} \left\| u^0(a-t) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&\quad + \int_{t^*}^\infty \left| e^{-\int_0^{t_*} m_{\bar{u}}(s,s+a-t_*)\mathrm{d}s} - e^{-\int_0^t m_{\bar{u}}(s,s+a-t)\mathrm{d}s} \right| \left\| U_{A[\bar{u}]}(t_*, 0) \right\|_{\mathcal{L}(E_\beta)} \left\| u^0(a-t_*) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&\quad + \int_{t^*}^\infty \left| e^{-\int_0^t m_{\bar{u}}(s,s+a-t)\mathrm{d}s} \right| \left\| \left[U_{A[\bar{u}]}(t_*, 0) - U_{A[\bar{u}]}(t, 0) \right] \, u^0(a-t_*) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&\quad + \int_{t^*}^\infty \left| e^{-\int_0^t m_{\bar{u}}(s,s+a-t)\mathrm{d}s} \right| \left\| U_{A[\bar{u}]}(t, 0) \right\|_{\mathcal{L}(E_\beta)} \left\| u^0(a-t_*) - u^0(a-t) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&\le c(T_0, R) \int_0^t \int_0^a |m_{\bar{u}}(s+t_*-a,s) - m_{\bar{u}}(s+t-a,s)| \, \mathrm{d}s \, a^{\bar{\mu}-\beta} \, \mathrm{d}a \\
&\quad + c(T_0, R) \int_0^t \left\| \left[U_{A[\bar{u}]}(t_*, t_*-a) - U_{A[\bar{u}]}(t, t-a) \right] B[u](t_*-a) \right\|_{E_\beta} \mathrm{d}a \\
&\quad + c(T_0, R) \int_0^t a^{\bar{\mu}-\beta} \, \|B[u](t_*-a) - B[u](t-a)\|_{E_\mu} \, \mathrm{d}a \\
&\quad + c(T_0, R) \int_t^{t^*} a^{\bar{\mu}-\beta} \, \mathrm{d}a \, + \, c(T_0, R) \int_0^{t_*-t} \|u^0(a)\|_{E_\beta} \, g(a) \, \mathrm{d}a \\
&\quad + c(T_0, R) \int_0^\infty \left| \int_0^{t_*} m_{\bar{u}}(s, s+a)\mathrm{d}s - \int_0^t m_{\bar{u}}(s, s+a+t_*-t)\mathrm{d}s \right| \left\| u^0(a) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&\quad + c(T_0, R) \int_0^\infty \left\| \left[U_{A[\bar{u}]}(t_*, 0) - U_{A[\bar{u}]}(t, 0) \right] u^0(a) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&\quad + c(T_0, R) \int_0^\infty \left\| u^0(a) - u^0(a-t+t_*) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\
&=: I + II + \ldots + VIII \,.
\end{aligned}$$

Now, as $|t - t_*| \to 0$ we clearly have $I + IV + V + VI \to 0$ due the Lebesgue Theorem (we obviously may assume $\beta \ge \mu$). Using $B[u] \in C(I_T, E_\mu)$, the density of the embedding $E_\beta \hookrightarrow E_\mu$, and the fact that the evolution system $U_{A[\bar{u}]}$ is uniformly strongly continuous on compact subsets of E_β, we also derive

that $II \to 0$. The continuity of $B[u]$ also entails $III \to 0$, while the strong continuity of $U_{A[\bar{u}]}$ on E_β ensures $VII \to 0$. Finally, $VIII \to 0$ holds since translations are strongly continuous. Therefore, $\Theta(u) \in C(I_T, \mathbb{E}_\beta)$.

Next observe that (3.7) implies

$$\left\| U_{A[\bar{u}]}(t,s) \right\|_{\mathcal{L}(E_\beta, E_\gamma)} \le c(T_0,R)(t-s)^{\beta-\gamma} + c_2 , \quad 0 \le s < t \le T , \quad u \in \mathcal{V}_T , \tag{3.12}$$

where

$$c_2 := \max_{0 \le s \le t \le T_0} \left\| U_{A[0]}(t,s) \right\|_{\mathcal{L}(E_\beta, E_\gamma)} .$$

In view of (3.1), (3.5), (3.9), (3.11), (3.12), and assumption (A_4) we deduce, for $u \in \mathcal{V}_T$ and $t \in I_T$, that

$$\begin{aligned} \|\Theta(u)(t)\|_{\mathbb{E}_\gamma} &\le c(T_0,R) \int_0^t \left\| U_{A[\bar{u}]}(t,t-a) \right\|_{\mathcal{L}(E_\mu, E_\gamma)} \|B[u](t-a)\|_{E_\mu} \, g(a) \, \mathrm{d}a \\ &\quad + c(T_0,R) \int_t^\infty \left\| U_{A[\bar{u}]}(t,0) \right\|_{\mathcal{L}(E_\beta, E_\gamma)} \left\| u^0(a-t) \right\|_{E_\beta} g(a) \, \mathrm{d}a \\ &\le c(T_0,R)\, t^{1+\bar{\mu}-\gamma} + c(T_0,R)\, t^{\beta-\gamma} \left\| u^0 \right\|_{\mathbb{E}_\beta} + c_2\, g_1 \, \|g\|_{L_\infty(0,T_0)} \left\| u^0 \right\|_{\mathbb{E}_\beta} . \end{aligned}$$

Since $\gamma < \beta$ we may choose $T = T(R) \in (0, T_0)$ sufficiently small to obtain

$$\|\Theta(u)(t)\|_{\mathbb{E}_\gamma} \le R + 1 , \quad t \in I_T , \quad u \in \mathcal{V}_T . \tag{3.13}$$

Moreover, writing for $u \in \mathcal{V}_T$ and $t \in I_T$

$$\begin{aligned} \overline{\Theta(u)}(t) = \int_0^\infty \Theta(u)(t,a)\, h(a) \, \mathrm{d}a &= \int_0^t e^{-\int_0^{t-a} m_{\bar{u}}(a+s,s)\mathrm{d}s} \, U_{A[\bar{u}]}(t,a)\, B[u](a)\, h(t-a) \, \mathrm{d}a \\ &\quad + \int_0^\infty e^{-\int_0^t m_{\bar{u}}(s,a+s)\mathrm{d}s} \, U_{A[\bar{u}]}(t,0)\, u^0(a)\, h(a+t) \, \mathrm{d}a \end{aligned} \tag{3.14}$$

and using the fact that, for $0 \le a \le t \le t_* \le T$,

$$\begin{aligned} \left\| U_{A[\bar{u}]}(t_*,a) - U_{A[\bar{u}]}(t,a) \right\|_{\mathcal{L}(E_\mu, E_\gamma)} &\le \left\| U_{A[\bar{u}]}(t_*,t) - U_{A[\bar{u}]}(t,t) \right\|_{\mathcal{L}(E_\beta, E_\gamma)} \left\| U_{A[\bar{u}]}(t,a) \right\|_{\mathcal{L}(E_\mu, E_\beta)} \\ &\le c(T_0,R)\, |t_* - t|^{\beta-\gamma} \, (t-a)^{\bar{\mu}-\beta} \end{aligned}$$

by (3.5) and (3.6), we derive from (A_1), (A_4), (3.1), (3.3), (3.5), (3.6), and (3.9) that, for $0 \le t \le t_* \le T$,

$$\begin{aligned}
&\left\|\overline{\Theta(u)}(t) - \overline{\Theta(u)}(t_*)\right\|_{\mathbb{E}_\gamma} \\
\le &\int_0^t \left| e^{-\int_0^{t_*-a} m_{\bar u}(a+s,s)\,\mathrm{d}s} - e^{-\int_0^{t-a} m_{\bar u}(a+s,s)\,\mathrm{d}s}\right| \left\|U_{A[\bar u]}(t_*,a)\right\|_{\mathcal{L}(E_\mu,E_\gamma)} \|B[u](a)\|_{E_\mu}\ h(t-a)\,\mathrm{d}a \\
&+ c(T_0,R)\int_0^t \left\|U_{A[\bar u]}(t_*,a) - U_{A[\bar u]}(t,a)\right\|_{\mathcal{L}(E_\mu,E_\gamma)} \|B[u](a)\|_{E_\mu}\ h(t-a)\,\mathrm{d}a \\
&+ c(T_0,R)\int_0^t \left\|U_{A[\bar u]}(t,a)\right\|_{\mathcal{L}(E_\mu,E_\gamma)} \|B[u](a)\|_{E_\mu}\ |h(t_*-a)-h(t-a)|\ \mathrm{d}a \\
&+ c(T_0,R)\int_t^{t_*} \left\|U_{A[\bar u]}(t_*,a)\right\|_{\mathcal{L}(E_\mu,E_\gamma)} \|B[u](a)\|_{E_\mu}\ |h(t-a)|\ \mathrm{d}a \\
&+ \int_0^\infty \left| e^{-\int_0^{t_*} m_{\bar u}(s,a+s)\,\mathrm{d}s} - e^{-\int_0^{t} m_{\bar u}(s,a+s)\,\mathrm{d}s}\right| \left\|U_{A[\bar u]}(t_*,0)\right\|_{\mathcal{L}(E_\beta,E_\gamma)} \left\|u^0(a)\right\|_{E_\beta}\ h(a+t_*)\,\mathrm{d}a \\
&+ c(T_0,R)\int_0^\infty \left\|U_{A[\bar u]}(t_*,0) - U_{A[\bar u]}(t,0)\right\|_{\mathcal{L}(E_\beta,E_\gamma)} \left\|u^0(a)\right\|_{E_\beta}\ h(a+t_*)\,\mathrm{d}a \\
&+ c(T_0,R)\int_0^\infty \left\|U_{A[\bar u]}(t,0)\right\|_{\mathcal{L}(E_\beta,E_\gamma)} \left\|u^0(a)\right\|_{E_\beta}\ |h(a+t_*)-h(a+t)|\ \mathrm{d}a \\
\le &c(T_0,R)\left\{ |t_*-t| + |t_*-t|^{\beta-\gamma} + \int_0^t (t-a)^{\bar\mu-\gamma}|h(t_*-a)-h(t-a)|\mathrm{d}a \right.\\
&\qquad\qquad \left. + |t_*-t|^{1+\bar\mu-\gamma} + |t_*-t| + |t_*-t|^{\beta-\gamma} + |t_*-t|^{\zeta}\right\}.
\end{aligned}$$

Taking into account that, due to (A_1),

$$\int_0^t (t-a)^{\bar\mu-\gamma}|h(t_*-a)-h(t-a)|\,\mathrm{d}a = \int_0^t a^{\bar\mu-\gamma}|h(t_*-t+a)-h(a)|\,\mathrm{d}a \le c(T_0)\,|t_*-t|^{\zeta}$$

and recalling the choice of θ, we may make $T = T(R) \in (0,T_0)$ smaller, if necessary, and conclude that

$$\left\|\overline{\Theta(u)}(t) - \overline{\Theta(u)}(t_*)\right\|_{\mathbb{E}_\gamma} \le |t_*-t|^{\theta}\,, \quad 0 \le t \le t_* \le T\,. \tag{3.15}$$

To prove that Θ is contractive, we observe that assumption (A_4) together with (3.1), (3.5), (3.7), (3.8), (3.9), and (3.10) imply that, for $u, u_* \in \mathcal{V}_T$, $0 \le t \le T \le T_0$, and for all $\xi \in [0,\beta]$,

$$\begin{aligned}
&\left\|\Theta(u)(t) - \Theta(u_*)(t)\right\|_{\mathbb{E}_\xi} \\
&\quad\le c(T_0,R)\int_0^t\int_0^a |m_{\bar u}(t-a+s,s) - m_{\bar u_*}(t-a+s,s)|\ \mathrm{d}s\ \left\|U_{A[\bar u]}(t,t-a)\right\|_{\mathcal{L}(E_\mu,E_\xi)} \\
&\qquad\qquad\qquad \times \|B[u](t-a)\|_{E_\mu}\ g(a)\,\mathrm{d}a \\
&\qquad + c(T_0,R)\int_0^t \left\|U_{A[\bar u]}(t,t-a) - U_{A[\bar u_*]}(t,t-a)\right\|_{\mathcal{L}(E_\mu,E_\xi)} \|B[u](t-a)\|_{E_\mu}\ g(a)\,\mathrm{d}a \\
&\qquad + c(T_0,R)\int_0^t \left\|U_{A[\bar u]}(t,t-a)\right\|_{\mathcal{L}(E_\mu,E_\xi)} \|B[u](t-a) - B[u_*](t-a)\|_{E_\mu}\ g(a)\,\mathrm{d}a
\end{aligned}$$

$$
\begin{aligned}
&+ c(T_0,R)\int_t^\infty \int_0^t |m_{\bar u}(s,a-t+s)\mathrm{d}s - m_{\bar u_*}(s,a-t+s)\mathrm{d}s|\ \big\|U_{A[\bar u]}(t,0)\big\|_{\mathcal{L}(E_\beta,E_\xi)}\\
&\qquad\qquad \times \big\|u^0(a-t)\big\|_{E_\beta}\, g(a)\,\mathrm{d}a\\
&+ c(T_0,R)\int_t^\infty \big\|U_{A[\bar u]}(t,0) - U_{A[\bar u_*]}(t,0)\big\|_{\mathcal{L}(E_\beta,E_\xi)}\ \big\|u^0(a-t)\big\|_{E_\beta}\, g(a)\,\mathrm{d}a\\
\le{}& c(T_0,R)\ \|u-u_*\|_{\mathcal{V}_T}\ \big\{t^{1+\bar\mu-\xi} + t^{1+\mu-\xi} + t^{1+\bar\mu-\xi} + t + t^{\beta-\xi}\big\}\ ,
\end{aligned}
$$

that is

$$
\big\|\Theta(u)(t) - \Theta(u_*)(t)\big\|_{\mathbb{E}_\xi} \le c(T_0,R)\left(t^{\bar\mu} + t^{\beta-\xi}\right)\ \|u-u_*\|_{\mathcal{V}_T}\ ,\quad t\in I_T\ . \tag{3.16}
$$

In particular, taking $\xi=\gamma<\beta$ we may choose $T=T(R)\in(0,T_0)$ sufficiently small such that

$$
\big\|\Theta(u)-\Theta(u_*)\big\|_{\mathcal{V}_T} \le \frac{1}{2}\ \|u-u_*\|_{\mathcal{V}_T}\ ,\quad u,u_*\in\mathcal{V}_T\ .
$$

Therefore, $\Theta:\mathcal{V}_T\to\mathcal{V}_T$ is a contraction provided $T=T(R)\in(0,T_0)$ is sufficiently small and hence possesses a unique fixed point, say u, in $\mathcal{V}_T\cap C(I_T,\mathbb{E}_\beta)$. Consequently,

$$
u(t,a) = \begin{cases} e^{-\int_0^a m_{\bar u}(s+t-a,s)\mathrm{d}s}\, U_{A[\bar u]}(t,t-a)\, B[u](t-a)\ , & 0\le a<t\ ,\\ e^{-\int_0^t m_{\bar u}(s,s+a-t)\mathrm{d}s}\, U_{A[\bar u]}(t,0)\, u^0(a-t)\ , & 0\le t<a\ ,\end{cases} \tag{3.17}
$$

for $0\le t\le T$ and $a>0$. An estimate similar to (3.13) combined with the strong continuity properties of the evolution system $U_{A[\bar u]}$ then warrants that

$$
u\in C_{v-\beta}\big((0,T],\mathbb{E}_v\big)\ ,\quad \beta\le v\le 1\ . \tag{3.18}
$$

In order to extend the just found solution $u\in C(I_T,\mathbb{E}_\beta)$, we choose

$$
R > \big(1+g_1\,\|g\|_{L_\infty(0,T_0)}\big)\ \|u\|_{C(I_T,\mathbb{E}_\beta)}\ \max_{0\le s\le t\le T_0}\|U_{A[0]}(t,s)\|_{\mathcal{L}(E_\beta,E_\gamma)}\ . \tag{3.19}
$$

similarly as in (3.1), and take now $\mathcal{V}_S$ to be

$$
\mathcal{V}_S := \big\{v\in C(I_S,\mathbb{E}_\gamma)\,;\ \|v(t)\|_{\mathbb{E}_\gamma}\le R+1\,,\ \|\bar v(t)-\bar v(t_*)\|_{E_\gamma}\le|t-t_*|^\theta\,,\ 0\le t,t_*\le S\,,\ v(0)=u(T)\big\}\ ,
$$

for $S>0$ with $T+S\le T_0$. Given $v\in\mathcal{V}_S$, we put

$$
V(t) := \begin{cases} u(t)\ , & 0\le t\le T\ ,\\ v(t-T)\ , & T\le t\le T+S\ ,\end{cases}
$$

and obtain $V\in C(I_{T+S},\mathbb{E}_\gamma)$ with $\|\bar V\|_{C^\theta(I_{T+S},E_\gamma)}\le R+2$. We then introduce

$$
\hat A[\bar v](t) := A[\bar V](t+T)\ ,\quad \hat B[v](t) := B[V](t+T)\ ,\quad \hat m_{\bar v}(t,a) := m(t+T,a,\bar v(t))
$$

for $t\in I_S$ and $a>0$. It follows from assumption (A_2) that

$$
\sigma+\hat A[\bar v]\in C\big(I_S,\mathcal{H}(E_1,E_0;\kappa,\omega)\big)
$$

and

$$
\hat A[\bar v]\in C^\rho\big(I_S,\mathcal{L}(E_1,E_0)\big)\quad\text{with}\quad \|\hat A[\bar v]\|_{C^\rho(I_S,\mathcal{L}(E_1,E_0))}\le c(T_0,R)
$$

for some σ, ω, κ, ρ depending on R and T_0. If also $v_*\in\mathcal{V}_S$, then

$$
\big\|\hat A[\bar v]-\hat A[\bar v_*]\big\|_{C(I_S,\mathcal{L}(E_1,E_0))}\le c(T_0,R)\,\|\bar v-\bar v_*\|_{C(I_S,E_\alpha)}\ .
$$

Hence $\hat A$ satisfies (2.2) and (2.3). Moreover, for $v,v_*\in\mathcal{V}_S$ we also have

$$
\big\|\hat B[v]-\hat B[v_*]\big\|_{C(I_S,E_\mu)}\le c(T_0,R)\,\|v-v_*\|_{C(I_S,\mathbb{E}_\alpha)}
$$

by (A_3), that is, $\hat{B}$ satisfies (2.4). Taking $S = S(R) > 0$ sufficiently small we deduce as before the existence of a function $v \in C(I_S, \mathbb{E}_\beta)$ with

$$v(t,a) \;=\; \begin{cases} e^{-\int_0^a \hat{m}_{\bar{v}}(s+t-a,s)\mathrm{d}s}\, U_{\hat{A}[\bar{v}]}(t,t-a)\, \hat{B}[v](t-a)\,, & 0 \le a < t\,, \\ e^{-\int_0^t \hat{m}_{\bar{v}}(s,s+a-t)\mathrm{d}s}\, U_{\hat{A}[\bar{v}]}(t,0)\, u(T,a-t)\,, & 0 \le t < a\,, \end{cases} \tag{3.20}$$

for $0 \le t \le S$ and $a > 0$. We then extend the function u by $w : I_{T+S} \to \mathbb{E}_\beta$ being defined as

$$w(t) := \begin{cases} u(t)\,, & 0 \le t \le T\,, \\ v(t-T)\,, & T \le t \le T+S\,. \end{cases}$$

Clearly, owing to $w\big|_{I_T} = u$ we infer $B[w]\big|_{I_T} = B[u]$ and $A[\bar{w}]\big|_{I_T} = A[\bar{u}]$ from assumptions (A_2), (A_3). Consequently, $U_{A[\bar{w}]}(t,s) = U_{A[\bar{u}]}(t,s)$ for $0 \le s \le t \le T$. Hence the function w still satisfies (3.17) in which u is replaced by w everywhere. Next, since

$$\hat{A}[\bar{v}](t-T) = A[\bar{w}](t)\,, \quad T \le t \le T+S\,,$$

we have by uniqueness

$$U_{\hat{A}[\bar{v}]}(t-T,s) = U_{A[\bar{w}]}(t,s+T)\,, \quad 0 \le s \le t-T \le S\,.$$

Furthermore,

$$\hat{B}[v](t-T-a) = B[w](t-a)\,, \quad T \le t \le T+S\,, \quad 0 \le t-T-a\,.$$

From these observations and using (3.17) and (3.20) it is then straightforward that

$$w(t,a) = v(t-T,a) \;=\; \begin{cases} e^{-\int_0^a m_{\bar{w}}(s+t-a,s)\mathrm{d}s}\, U_{A[\bar{w}]}(t,t-a)\, B[w](t-a)\,, & 0 \le a < t\,, \\ e^{-\int_0^t m_{\bar{w}}(s,s+a-t)\mathrm{d}s}\, U_{A[\bar{w}]}(t,0)\, u^0(a-t)\,, & 0 \le t < a\,, \end{cases}$$

for $T \le t \le T+S$. Therefore, we may extend u to a unique maximal generalized solution u in $C(J, \mathbb{E}_\beta)$ satisfying (3.17) for $t \in J$ and $a > 0$. Clearly, the maximal interval of existence, J, is open in $[0,\infty)$. If (2.7) holds true, then (3.19) and the above extension procedure yield $J = \mathbb{R}^+$. Obviously, (3.18) holds for any $T \in \dot{J}$. Moreover, proceeding as in (3.15) shows that for any function $\tilde{h}$ satisfying (A_1) we have

$$\int_0^\infty u(\cdot,a)\, \tilde{h}(a)\, \mathrm{d}a \in C^{\zeta \wedge (\gamma-\beta)}(J, E_\gamma)$$

for $\gamma \in [\alpha, \beta)$. Taking $h \equiv 1$, (3.14) reads

$$\begin{aligned} \int_0^\infty u(t,a)\, \mathrm{d}a \;&=\; \int_0^t U_{A[\bar{u}]}(t,a)\, e^{-\int_0^{t-a} m_{\bar{u}}(a+s,s)\mathrm{d}s}\, B[u](a)\, \mathrm{d}a \\ &\quad + U_{A[\bar{u}]}(t,0) \int_0^\infty e^{-\int_0^t m_{\bar{u}}(s,a+s)\mathrm{d}s}\, u^0(a)\, \mathrm{d}a\,. \end{aligned}$$

The right hand side is clearly differentiable with respect to t and, owing to (3.17) and (3.18) with $\upsilon = 1$, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty u(t,a)\, \mathrm{d}a = -A[\bar{u}](t) \int_0^\infty u(t,a)\, \mathrm{d}a + B[u](t) - \int_0^\infty m_{\bar{u}}(t,a)\, u(t,a)\, \mathrm{d}a$$

from which we conclude (2.6) by invoking [1, II.Thm.1.2.2]. This proves Theorem 2.2

4. Proof of Further Properties

For the remainder of this section, we fix $u^0 \in \mathbb{E}_\beta$ and let $u = u(\cdot; u^0) \in C(J, \mathbb{E}_\beta) \cap C_{\upsilon-\beta}((0,T], \mathbb{E}_\upsilon)$ for $T \in \dot{J}$ and $\beta \le \upsilon \le 1$ denote the unique maximal generalized solution to (1.1)-(1.4) on $J = J(u^0)$ corresponding to u^0.

4.1. **Proof of Corollary 2.4.** We use the notation as in the proof of Theorem 2.2. Given $u^0 \in \mathbb{E}_\beta$ it is clear that we may choose $\delta > 0$ such that (3.1) holds true if u^0 therein is replaced by any $u^0_* \in \mathbb{E}_\beta$ with $\|u^0 - u^0_*\|_{\mathbb{E}_\beta} \le \delta$. Therefore, the proof of Theorem 2.2 shows that there are solutions $u = u(\cdot; u^0)$ and $u_* = u(\cdot; u^0_*)$ both belonging to $\mathcal{V}_T$, where $T = T(R) \in (0, T_0)$ is sufficiently small. As in (3.16) we obtain for any $\bar\mu \in (0,\mu)$, $\xi \in [0,\beta]$, and $t \in [0,T]$

$$\begin{aligned} \|u(t) - u_*(t)\|_{\mathbb{E}_\xi} \,&\le\, c(T_0,R)\, \big(t^{\bar\mu} + t^{\beta-\xi}\big)\, \|u-u_*\|_{\mathcal{V}_T} \\ &\qquad + c(T_0,R) \int_t^\infty \big\|U_{A[\bar u_*]}(t,0)\big\|_{\mathcal{L}(E_\beta,E_\xi)} \big\|u^0(a-t) - u^0_*(a-t)\big\|_{E_\beta}\, g(a)\,\mathrm{d}a \\ &\le\, c(T_0,R)\, \big(t^{\bar\mu} + t^{\beta-\xi}\big)\, \|u-u_*\|_{\mathcal{V}_T} \,+\, c(T_0,R)\, \big\|u^0 - u^0_*\big\|_{\mathbb{E}_\beta}\ . \end{aligned}$$

Hence, by taking $\xi = \gamma < \beta$ and making $T = T(R) \in (0,T_0)$ smaller if necessary, we first deduce

$$\|u-u_*\|_{\mathcal{V}_T} \,\le\, c(T_0,R)\, \big\|u^0 - u^0_*\big\|_{\mathbb{E}_\beta}$$

and then, choosing $\xi = \beta$,

$$\|u(t) - u_*(t)\|_{\mathbb{E}_\beta} \,\le\, c(T_0,R)\, \big\|u^0 - u^0_*\big\|_{\mathbb{E}_\beta}\ ,\quad t \in [0,T]\ ,$$

whence the claim of Corollary 2.4.

4.2. **Proof of Proposition 2.5.** To establish Proposition 2.5 we use the properties of evolution systems [1]

$$\frac{\partial}{\partial t} U_{A[\bar u]}(t,s)w \,=\, -A[\bar u](t) U_{A[\bar u]}(t,s)w\ ,\quad 0 \le s < t \in J\ ,\quad w \in E_0\ ,$$

and

$$\frac{\partial}{\partial s} U_{A[\bar u]}(t,s)v \,=\, U_{A[\bar u]}(t,s) A[\bar u](s)v\ ,\quad 0 \le s < t \in J\ ,\quad v \in E_1\ .$$

Then, due to (2.8) and (2.9), it follows from (3.17) that, for $t \in \dot J$ and $a > 0$ with $a \neq t$,

$$\begin{aligned} \partial_t u(t,a) \,&= \mathbf{1}_{[a<t]}(t,a) \Big\{ e^{-\int_{t-a}^t m_{\bar u}(s,s+a-t)\,\mathrm{d}s} U_{A[\bar u]}(t,t-a) \big(A[\bar u](t-a) + \partial_t\big)\, B[u](t-a) \\ &\qquad + \Big(-m_{\bar u}(t,a) + m_{\bar u}(t-a,0) + \int_{t-a}^t \partial_2 m_{\bar u}(s,s+a-t)\,\mathrm{d}a - A[\bar u](t) \Big)\, u(t,a) \Big\} \\ &\quad + \mathbf{1}_{[a>t]}(t,a) \Big\{ \Big(-m_{\bar u}(t,a) + \int_0^t \partial_2 m_{\bar u}(s,s+a-t)\,\mathrm{d}a - A[\bar u](t) \Big)\ u(t,a) \\ &\qquad - e^{-\int_0^t m_{\bar u}(s,s+a-t)\,\mathrm{d}s} U_{A[\bar u]}(t,0) \partial_a u^0(a-t) \Big\} \end{aligned}$$

if $m \in C^{0,1}(J \times \mathbb{R}^+)$ and similarly

$$\begin{aligned} \partial_a u(t,a) \,&= \mathbf{1}_{[a<t]}(t,a) \Big\{ \Big(-m_{\bar u}(t-a,0) - \int_{t-a}^t \partial_2 m_{\bar u}(s,s+a-t)\,\mathrm{d}a \Big)\ u(t,a) \\ &\qquad + e^{-\int_{t-a}^t m_{\bar u}(s,s+a-t)\,\mathrm{d}s} U_{A[\bar u]}(t,t-a) \big(-A[\bar u](t-a) - \partial_t \big)\, B[u](t-a) \Big\} \\ &\quad + \mathbf{1}_{[a>t]}(t,a) \Big\{ \Big(\int_0^t \partial_2 m_{\bar u}(s,s+a-t)\,\mathrm{d}a - A[\bar u](t) \Big)\ u(t,a) \\ &\qquad + e^{-\int_0^t m_{\bar u}(s,s+a-t)\,\mathrm{d}s} U_{A[\bar u]}(t,0) \partial_a u^0(a-t) \Big\}\ . \end{aligned}$$

Taking into account the particular form of u in (3.17), the assumptions on B and u^0, and the continuity properties of $U_{A[\bar u]}$ we deduce that u indeed possesses the regularity (2.10), (2.11) and satisfies

$$\big(\partial_t + \partial_a\big) u(t,a) = -A[\bar u](t) u(t,a) - m_{\bar u}(t,a) u(t,a) \quad \text{in} \quad E_0$$

for $t \in \dot{J}$ and $a > 0$ with $t \neq a$. Clearly,

$$u(t,0) = B[u](t)\ , \quad t \in \dot{J}\ , \qquad u(0,a) = u^0(a)\ , \quad a > 0\ ,$$

where both equations hold in E_0. This proves Proposition 2.5.

4.3. **Proof of Proposition 2.6.** Suppose the assumptions of Proposition 2.6. Replacing $\mathcal{V}_T$ in the proof of Theorem 2.2 by the closed metric space

$$\mathcal{V}_T^+ := \left\{ v \in C(I_T, \mathbb{E}_\gamma^+)\, ;\, \|v(t)\|_{\mathbb{E}_\gamma} \le R+1\ ,\ \|\bar{v}(t) - \bar{v}(t_*)\|_{E_\gamma} \le |t-t_*|^\theta\ ,\ 0 \le t, t_* \le T \right\}\ ,$$

and using the fact that the resolvent positivity of the operator A implies

$$U_{A[\bar{v}]}(t,s) : E_0^+ \to E_0^+\ , \quad 0 \le s \le t \le T\ , \quad v \in \mathcal{V}_T^+\ ,$$

by [1, II.6.4], it follows from the assumptions on B that the map Θ, introduced in the proof of Theorem 2.2, is a contraction from $\mathcal{V}_T^+$ into itself (provided T is chosen sufficiently small). This then readily gives Proposition 2.6.

4.4. **Proof of Proposition 2.7.** Suppose the assumptions of Proposition 2.7 and let $T > 0$ be arbitrary. Observe that we may assume without loss of generality in (2.14) that $\vartheta \le \beta$. Then (2.12), (2.13) in combination with [1, II.Lem.5.1.3] ensure the existence of a constant $c_3 = c_3(T) > 0$ such that

$$\left\|U_{A[\bar{u}]}(t,s)\right\|_{\mathcal{L}(E_\beta)} + (t-s)^{\beta-\vartheta/2} \left\|U_{A[\bar{u}]}(t,s)\right\|_{\mathcal{L}(E_\vartheta, E_\beta)} \le c_3\ , \quad 0 \le s < t \in J_T\ . \tag{4.1}$$

Introducing $z \in C(J_T)$ by

$$z(\tau) := \max_{0 \le t \le \tau} \|u(t)\|_{\mathbb{E}_\beta}\ , \quad \tau \in J_T\ ,$$

it follows from (2.14), (4.1), and the assumption that m is non-negative or bounded

$$\begin{aligned}
\|u(t)\|_{\mathbb{E}_\beta} &\le c(T) \int_0^t \left\|U_{A[\bar{u}]}(t,t-a)\right\|_{\mathcal{L}(E_\vartheta,E_\beta)} \|B[u](t-a)\|_{E_\vartheta}\ g(a)\, \mathrm{d}a \\
&\qquad + c(T) \int_t^\infty \left\|U_{A[\bar{u}]}(t,0)\right\|_{\mathcal{L}(E_\beta)} \left\|u^0(a-t)\right\|_{E_\beta}\ g(a)\, \mathrm{d}a \\
&\le c(T) c_1 c_3 \|g\|_{L_\infty(J_T)} \int_0^t a^{-\beta+\vartheta/2} \big(1 + z(t-a)\big)\, \mathrm{d}a + c(T) g_1 c_3 \|g\|_{L_\infty(J_T)} \left\|u^0\right\|_{\mathbb{E}_\beta}
\end{aligned}$$

for $t \in J_T$ and thus, since z is non-decreasing,

$$z(\tau) \le c(T) \left(1 + \int_0^\tau (\tau-a)^{-\beta+\vartheta/2} z(a)\, \mathrm{d}a \right)\ , \quad \tau \in J_T\ .$$

Due to $\beta - \vartheta/2 < 1$, Gronwall's inequality implies (2.7), hence $J = \mathbb{R}^+$.

4.5. **Proof of Remark 2.8.** We note that if (2.14) is replaced with (2.15), we clearly may assume that $\upsilon \in [\beta, 1)$. Then, as in the proof of Proposition 2.7 we obtain from (2.15) and the analogue of (4.1)

$$\begin{aligned}
\|u(t)\|_{\mathbb{E}_\upsilon} &\le c(T) c_2 c_3 \|g\|_{L_\infty(J_T)} \int_0^t a^{-\upsilon+\vartheta/2} \left(1 + \|u(t-a)\|_{\mathbb{E}_\upsilon}\right) \mathrm{d}a \\
&\qquad + c(T) g_1 c_2 \|g\|_{L_\infty(J_T)} \left\|u^0\right\|_{\mathbb{E}_\beta} t^{-\upsilon+\beta/2} \\
&\le c(T) \left(1 + t^{-\upsilon+\beta/2} + \int_0^t (t-a)^{-\upsilon+\vartheta/2} \|u(a)\|_{\mathbb{E}_\upsilon}\, \mathrm{d}a \right)
\end{aligned}$$

for $t \in \dot{J}_T$. Applying the singular Gronwall inequality [1, II.Cor.3.3.2], we deduce (2.7).

5. Applications

We give examples of problems to which the results of Section 2 may be applied. First we provide some conditions intended to simplify the verification of assumption (A_2) and (2.12), (2.13).

5.1. **General Remarks.** We show that if A has a particular form, then assumption (A_2) is rather easy to verify in concrete applications. This result, in particular, applies to the case when A depends locally with respect to time on $\bar{u}$.

More precisely, we assume that A is of the form

$$A[\bar{z}](t) = A_0\big(t, \Phi(\bar{z})(t)\big)\ , \tag{5.1}$$

where

$$A_0 \in C_b^{\varrho,1-}\big(\mathbb{R}^+ \times F_0, \mathcal{H}(E_1, E_0)\big) \tag{5.2}$$

for some Banach space F_0 and $\varrho \in (0,1)$. That is, given any $R > 0$ there exists $c(R) > 0$ such that

$$\|A_0(t,z) - A_0(t_*, z_*)\|_{\mathcal{H}(E_1,E_0)} \ \le\ c(R)\left(|t - t_*|^\varrho + \|z - z_*\|_{F_0}\right)$$

for $t, t_* \in [0,R]$ and $z, z_* \in F_0$ with $\|z\|_{F_0}, \|z_*\|_{F_0} \le R$.

Given another Banach space F_1 with $F_1 \hookrightarrow F_0$, the function Φ is supposed to satisfy the following conditions (for some $\alpha \in [0,1)$):

(A_5) Given $T_0, R > 0$ and $\theta \in (0,1)$ there are numbers $\rho \in (0,1)$ and $c_4 > 0$ (depending on T_0, R, and θ) such that, for each $T \in (0, T_0)$, the function Φ maps $C^\theta(I_T, E_\alpha)$ into $C^\rho(I_T, F_1)$ and satisfies

$$\|\Phi(\bar{z})(t) - \Phi(\bar{z})(t_*)\|_{F_1} \ \le c_4\,|t - t_*|^\rho$$

and

$$\|\Phi(\bar{z})(t) - \Phi(\bar{z}_*)(t)\|_{F_1} \ \le\ c_4\,\|\bar{z} - \bar{z}_*\|_{C(I_T,E_\alpha)}$$

for $t, t_* \in I_T$ and all $\bar{z}, \bar{z}_* \in C^\theta(I_T, E_\alpha)$ with $\|\bar{z}\|_{C^\theta(I_T,E_\alpha)} \le R$ and $\|\bar{z}_*\|_{C^\theta(I_T,E_\alpha)} \le R$. Moreover, if $0 < T < S$ and $\bar{z}, \bar{z}_* \in C(I_S, E_\alpha)$ with $\bar{z}\big|_{I_T} = \bar{z}_*\big|_{I_T}$, then $\Phi(\bar{z})\big|_{I_T} = \Phi(\bar{z}_*)\big|_{I_T}$.

Then we have:

Proposition 5.1. *Suppose that the embedding $F_1 \hookrightarrow F_0$ is compact and let the operator A be of the form* (5.1) *with A_0 satisfying* (5.2) *and Φ satisfying assumption (A_5). Then A satisfies assumption (A_2).*

Proof. Given $T_0, R > 0$ and $\theta \in (0,1)$ it follows from assumption (A_5) that there exists a bounded set $M \subset F_1$ such that $\Phi(\bar{z})(t) \in M$ for all $0 \le t \le T \le T_0$ and $\bar{z} \in C^\theta(I_T, E_\alpha)$ with $\|\bar{z}\|_{C^\theta(I_T,E_\alpha)} \le R$. Due to the compactness of the embedding $F_1 \hookrightarrow F_0$ we deduce that M is relatively compact in F_0 and so is $A_0\big([0,T_0] \times M\big)$ in $\mathcal{H}(E_1, E_0)$ by continuity. Hence, [1, I.Cor.1.3.2] ensures the existence of numbers $\kappa \ge 1$ and $\omega > 0$ such that $A_0\big([0,T_0] \times M\big) \subset \mathcal{H}(E_1, E_0; \kappa, \omega)$. But then (A_5), (5.1), (5.2), and the continuous embedding $F_1 \hookrightarrow F_0$ readily imply (A_2). $\square$

It is worthwhile to point out that assumption (A_5) is trivially satisfied if Φ is the identity. Therefore, assumption (A_2) holds for operators A depending locally with respect to time on $\bar{z}$:

Corollary 5.2. *Suppose that the embedding $E_1 \hookrightarrow E_0$ is compact and let the operator A be of the form $A[\bar{z}](t) = A_0\big(t, \bar{z}(t)\big)$, where $A_0 \in C_b^{\varrho,1-}\big(\mathbb{R}^+ \times E_\sigma, \mathcal{H}(E_1, E_0)\big)$ for some $\varrho \in (0,1)$ and $\sigma \in [0,1)$. Then A satisfies assumption (A_2) for any $\alpha \in (\sigma, 1]$.*

Proof. It just remains to observe that the embedding $F_1 := E_\alpha \hookrightarrow E_\sigma =: F_0$ is compact according to [1, I.Thm.2.11.1] since $\sigma < \alpha$ and due to the choice of admissible interpolation functors $(\cdot,\cdot)_\theta$. $\square$

If A is of the form (5.1), then also the conditions (2.12), (2.13) for global existence are simpler to verify. Thus we consider again the unique maximal generalized solution $u = u(\cdot; u^0) \in C(J, \mathbb{E}_\beta)$ to (1.1)-(1.4) on $J = J(u^0)$ corresponding to u^0 as provided by Theorem 2.2.

Corollary 5.3. *Suppose that the embedding $F_1 \hookrightarrow F_0$ is compact and let the operator A be of the form* (5.1) *with A_0 satisfying* (5.2)*. Let Φ satisfy assumption (A_5) and suppose that for each $T > 0$ there exist numbers $\rho \in (0,1)$ and $c_5(T) > 0$ such that the solution u to* (1.1)-(1.4) *satisfies*

$$\|\Phi(\bar{u})(t) - \Phi(\bar{u})(t_*)\|_{F_1} \le c_5(T)\,|t - t_*|^\rho\ , \quad t, t_* \in J_T := J \cap [0,T]\ . \tag{5.3}$$

Then (2.12) *and* (2.13) *hold.*

Proof. Since (5.3) in particular means that $\Phi(\bar{u})(J_T)$ is bounded in F_1, (2.13) is immediate from (5.2). Analogously as in the proof of Lemma 5.1, condition (2.12) is a consequence of [1, I.Cor.1.3.2] and the compact embedding $F_1 \hookrightarrow F_0$. □

5.2. **Applications.** Since the following exemplary problems were studied elsewhere (except for the first one), we do not go too much into the details. Clearly, the results of Section 2 do not restrict to the examples presented herein.

For the remainder we fix a bounded subset Ω of $\mathbb{R}^n$, $n \le 3$, with smooth boundary $\partial\Omega$. Even though we may incorporate general time-dependent second order elliptic operators on Ω subject to suitable boundary conditions, we restrict ourselves for the sake of simplicity to time-independent operators in divergence form, that is, operators of the form

$$A_0(z)w := -\nabla_x \cdot \big(D(z)\nabla_x w\big) \tag{5.4}$$

subject to, e.g., Neumann conditions on $\partial\Omega$. Here, the function D is supposed to satisfy

$$D \in C^{2-}(\mathbb{R})\ , \qquad D(z) \ge d_0 > 0\ , \quad z \in \mathbb{R}\ , \tag{5.5}$$

for some number d_0. Introducing for $p \in (1,\infty)$ and $\theta \ge 0$ the Sobolev spaces (including Neumann boundary conditions)

$$W^{2\theta}_{p,\mathcal{B}} := \begin{cases} \big\{u \in W^{2\theta}_p(\Omega)\,;\, \partial_\nu u = 0\big\}\ , & 2\theta > 1 + 1/p\ , \\ W^{2\theta}_p(\Omega)\ , & 0 \le 2\theta \le 1 + 1/p\ , \end{cases}$$

we obtain that

$$\text{the embedding} \quad W^2_{p,\mathcal{B}} \hookrightarrow L_p \quad \text{is compact} \tag{5.6}$$

and

$$E_{1/2} := [L_p, W^2_{p,\mathcal{B}}]_{1/2} = W^1_{p,\mathcal{B}}\ , \qquad E_\theta := (L_p, W^2_{p,\mathcal{B}})_{\theta,p} = W^{2\theta}_{p,\mathcal{B}}\ , \quad 2\theta \in (0,2)\setminus\{1, 1+1/p\}\ , \tag{5.7}$$

where the equality is up to equivalent norms and where $[\cdot,\cdot]_{1/2}$ and $(\cdot,\cdot)_{\theta,p}$ are the complex and real interpolation functors, respectively, all of which are admissible. Moreover,

$$A_0 \in C^{1-}_b\big(C^1(\bar{\Omega}), \mathcal{H}(W^2_{p,\mathcal{B}}, L_p)\big) \tag{5.8}$$

due to (5.5), and

$$A_0(z) \quad \text{is resolvent positive for } z \in C^1(\bar{\Omega})\ . \tag{5.9}$$

We assume that a non-negative function $b \in C^{2-}(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$ is given that satisfies, for any $T, R > 0$,

$$|D^k_3 b(t,a,z) - D^k_3 b(t,a,z_*)| \le c(T,R)\,g(a)\,|z - z_*|\ , \tag{5.10}$$

for $t \in [0,T]$, $a \ge 0$, $|z|, |z_*| \le R$, and $k = 0,1$, where $g \in L_{\infty,loc}(\mathbb{R}^+)$ satisfies (2.1). For simplicity we also assume that b is bounded, that is,

$$b(t,a,z) \le c(T)\,g(a)\ , \tag{5.11}$$

for $t \in [0,T]$, $a \ge 0$, and all $z \in \mathbb{R}$ (this is merely needed to guarantee that solutions exists globally in the subsequent examples). Thus, it follows from (the proof of) [18, Lem.2.7] that

$$\|b(t,a,\bar{z}) - b(t,a,\bar{z}_*)\|_{W^{2\bar{\alpha}}_p} \le c(T,R)\,g(a)\,\|\bar{z} - \bar{z}_*\|_{W^{2\alpha}_p}\ , \tag{5.12}$$

$$\|b(t,a,\bar{z})\|_{W^{2\bar{\alpha}}_p} \le c(T,R)\,g(a)\ , \tag{5.13}$$

for $t \in [0,T]$, $\bar z, \bar z_* \in W_p^{2\alpha}$ with $\|\bar z\|_{W_p^{2\alpha}}, \|\bar z_*\|_{W_p^{2\alpha}} \le R$, provided that $n/p < 2\bar\alpha < 2\alpha$. Also note that there is $2\mu \in (n/p, 2\bar\alpha)$ such that (see [3])

$$\text{pointwise multiplication} \quad W_p^{2\bar\alpha} \times W_p^{2\alpha} \to W_p^{2\mu} \quad \text{is continuous} \ . \tag{5.14}$$

We put

$$\mathbb{W}_{p,\mathcal{B}}^{2\theta} := L_1\big(\mathbb{R}^+, W_{p,\mathcal{B}}^{2\theta}, g(a)\mathrm{d}a\big) \ .$$

Then $z \in \mathbb{W}_{p,\mathcal{B}}^{2\theta}$ is non-negative if $z \in \mathbb{W}_{p,\mathcal{B}}^{2\theta} \cap L_1\big(\mathbb{R}^+, L_p^+, g(a)\mathrm{d}a\big)$ with L_p^+ denoting the positive cone of $L_p = L_p(\Omega)$. Let

$$m \in C(\mathbb{R}^+) \quad \text{with} \quad m \ge 0 \ . \tag{5.15}$$

5.2.1. *Birth boundary conditions with delay.* We consider a model with history-dependent birth rate as investigated in [7] for the spatially homogeneous case:

$$\partial_t u + \partial_a u = \mathrm{div}_x\big(D(\bar u(t,x))\, \nabla_x u\big) - m(a)\, u \ , \qquad (t,a,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \Omega \ , \tag{5.16}$$

$$u(t,0,x) = \int_0^\infty b\left(t, a, \int_{-\tau}^0 \bar u(t+\sigma)\mathrm{d}\sigma\right) u(t,a)\, \mathrm{d}a \ , \qquad (t,x) \in \mathbb{R}^+ \times \Omega \ , \tag{5.17}$$

$$u(s,a,x) = F(s,a,x) \ , \qquad (s,a,x) \in [-\tau,0] \times \mathbb{R}^+ \times \Omega \ , \tag{5.18}$$

$$\partial_\nu u(t,a,x) = 0 \ , \qquad (t,a,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \partial\Omega \ , \tag{5.19}$$

$$\bar u(t,x) = \int_0^\infty u(t,a,x)\, \mathrm{d}a \ , \qquad (t,x) \in [-\tau,\infty) \times \Omega \ , \tag{5.20}$$

where $\tau > 0$ is the maximal delay.

Proposition 5.4. *Let* $g \equiv 1$ *and suppose* (5.4), (5.5), (5.10), (5.11), (5.15). *Let* $F \in C([-\tau,0], \mathbb{W}_{p,\mathcal{B}}^{2\beta})$ *be non-negative, where* $1 + n/p < 2\beta \le 2$. *Then* (5.16)-(5.20) *admit a unique non-negative generalized solution* $u \in C(\mathbb{R}^+, \mathbb{W}_{p,\mathcal{B}}^{2\beta})$ *with* $\bar u \in C^1((0,\infty), L_p) \cap C((0,\infty), W_{p,\mathcal{B}}^2)$.

Proof. We merely sketch the proof. Extending a given function $u \in C(\mathbb{R}^+, \mathbb{W}_{p,\mathcal{B}}^{2\beta})$ by

$$u(t,\cdot,\cdot) := \begin{cases} u(t,\cdot,\cdot) \ , & t \ge 0 \ , \\ F(t,\cdot,\cdot) \ , & t \in [-\tau,0) \ , \end{cases}$$

and defining

$$A[\bar u](t) := A_0(\bar u(t)) \ , \qquad B[u](t) := \int_0^\infty b\left(t, a, \int_{-\tau}^0 \bar u(t+\sigma)\mathrm{d}\sigma\right) u(t,a)\, \mathrm{d}a \ ,$$

equations (5.16)-(5.20) may be written in the form (1.1)-(1.4) with $u^0 := F(0,\cdot,\cdot)$. Then (A_2) is a consequence of (5.6)-(5.8) and Corollary 5.2 by observing that

$$W_{p,\mathcal{B}}^{2\beta} \hookrightarrow W_{p,\mathcal{B}}^{2\alpha} \hookrightarrow C^1(\bar\Omega) \ , \quad 1 + n/p < 2\alpha < 2\beta \ , \tag{5.21}$$

while (A_3) follows from (5.12) and (5.14). Therefore, local existence of a non-negative generalized maximal solution $u \in C(J, \mathbb{W}_{p,\mathcal{B}}^{2\beta})$ on some maximal interval J is immediate from Theorem 2.2, Proposition 2.6, (5.9), and (5.15). Next note that $\bar u \in C^1(\dot J, L_p) \cap C(\dot J, W_{p,\mathcal{B}}^2)$ solves

$$\partial_t \bar u - \nabla_x \cdot \big(D(\bar u)\nabla_x \bar u\big) = -\int_0^\infty m(a) u(t,a)\mathrm{d}a + B[u](t) =: f(t,x)$$

in $\dot J_T \times \Omega$, with $J_T := J \cap [0,T]$ for $T > 0$ fixed. Since $B[u] \in C(J_T, W_p^{2\mu})$ by (5.14) and $W_p^{2\mu} \hookrightarrow C(\bar\Omega)$ we have $f \in C(J_T \times \bar\Omega)$. From (5.11) and the maximum principle we first obtain $\bar u \in L_\infty(J_T, L_\infty(\Omega))$ and then $f \in BC(J_T \times \bar\Omega)$. Hence, [2, Thm.4.2, Rem.4.3] entail that $\bar u : J_T \to C^{1+\epsilon}(\bar\Omega)$ is bounded and uniformly Hölder continuous with $\epsilon > 0$. Since the embedding $F_1 := C^{1+\epsilon}(\bar\Omega) \hookrightarrow C^1(\bar\Omega) =: F_0$ is compact, we deduce (2.12) and (2.13) from Corollary 5.3, while (2.14) is obvious. Proposition 2.7 then gives $J = \mathbb{R}^+$. □

5.2.2. *A tumor invasion model.* The following *haptotaxis* model describes the invasion of tumor cells (with density u) into the surrounding tissue along gradients of bound cell adhesion molecules (with density f) that are contained in the extracellular matrix. The cells produce a matrix degradative enzyme with density v. The model was studied in detail in [17, 18], and we just recall a very simple version:

$$\begin{aligned}
\partial_t u + \partial_a u \ &= \ \mathrm{div}_x\big(D(f)\, \nabla_x u - u\chi(f)\nabla_x f\big) \ - \ m(a)\, u\ , && (t,a,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \Omega\ , && (5.22)\\
\partial_t f \ &= \ -vf\ , && (t,x) \in \mathbb{R}^+ \times \Omega\ , && (5.23)\\
\partial_t v \ &= \ \Delta_x v + \bar{u} - v\ , && (t,x) \in \mathbb{R}^+ \times \Omega\ , && (5.24)\\
u(t,0,x) \ &= \ \int_0^\infty b(t,a,\bar{u}(t))\, u(t,a)\, \mathrm{d}a\ , && (t,x) \in \mathbb{R}^+ \times \Omega\ , && (5.25)\\
u(0,a,x) \ &= \ u^0(a,x)\ ,\ f(0,x) \ = \ f^0(x)\ ,\ v(0,x) \ = \ v^0(x)\ , && (a,x) \in \mathbb{R}^+ \times \Omega\ , && (5.26)\\
\partial_\nu v \ &= \ D(f)\partial_\nu u - u\chi(f)\partial_\nu f \ = \ 0\ , && (t,a,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \partial\Omega\ , && (5.27)\\
\bar{u}(t,x) \ &= \ \int_0^\infty u(t,a,x)\, \mathrm{d}a\ , && (t,x) \in \mathbb{R}^+ \times \Omega\ . && (5.28)
\end{aligned}$$

If χ is smooth and D satisfies (5.5), we obtain for

$$A_1(f) := [w \mapsto w\chi(f)\nabla_x f]$$

that

$$A_0 + A_1 \in C_b^{1-}\big(W^2_{p,\mathcal{B}}, \mathcal{H}(W^2_{p,\mathcal{B}}, L_p)\big)\ , \quad p > n\ . \tag{5.29}$$

Given initial values $(f^0, v^0) \in W^{2+\tau}_{p,\mathcal{B}} \times W^{2\tau}_{p,\mathcal{B}}$ with $\tau > 0$ and a suitable function $\bar{u}$, we first solve (5.24) for v and plug the result into equation (5.23). It follows from [16, Lem.2.1] and [18, Lem.2.6] that

$$\Phi(\bar{u}) := f \quad \text{satisfies } (A_5) \text{ with } F_1 := W^{2+\epsilon}_{p,\mathcal{B}},\ \epsilon \in (0,\tau), \text{ and any } \alpha \in [0,1)\ . \tag{5.30}$$

We then recall the result of [18]:

Proposition 5.5. *Let* $g \equiv 1$ *and suppose* (5.5), (5.10), (5.11), *and* (5.15). *Let* χ *be a smooth function. Let* $p > n$, $\tau > 0$, *and* $2\beta \in (n/p, 2) \setminus \{1 + 1/p\}$. *Then, given non-negative initial values*

$$(f^0, v^0, u^0) \in X := W^{2+\tau}_{p,\mathcal{B}} \times W^{2\tau}_{p,\mathcal{B}} \times \mathbb{W}^{2\beta}_{p,\mathcal{B}}$$

there exists a unique non-negative solution $(f, v, u) \in C(\mathbb{R}^+, X)$ *to* (5.22)-(5.28), f *and* v *being classical solutions to the corresponding equations. Moreover,* $\bar{u} \in C^1(\mathbb{R}^+, L_p) \cap C(\mathbb{R}^+, W^2_{p,\mathcal{B}})$.

Proof. We simply outline the main ideas of the proof of Proposition 5.5 and refer to [18] for details. First, local existence is immediate from Theorem 2.2, Corollary 5.1, (5.12), (5.15), (5.13), (5.29), and (5.30). Given $T > 0$ one can prove by a bootstrapping argument that $f = \Phi(\bar{u}) : J_T \to W^2_{p,\mathcal{B}}$ is uniformly Hölder continuous and bounded (see [18, Eq.(3.26)]), whence (2.13) follows from (5.29). In particular, since $f(J_T)$ is bounded in $W^2_{p,\mathcal{B}}$ and the embedding $W^2_{p,\mathcal{B}} \hookrightarrow C^1(\bar{\Omega})$ is compact, we derive from [1, I.Cor.1.3.2] that $A_0(f(J_T))$ is a subset of $\mathcal{H}(W^2_{p,\mathcal{B}}, L_p; \kappa, \omega)$ for some $\kappa \geq 1$, $\omega > 0$. Considering $A_1(f)$ as a perturbation of $A_0(f)$, we deduce (2.12) using [1, I.Thm.1.3.1(b)]. Thus $J = \mathbb{R}^+$ by Proposition 2.7 since (2.14) is obvious.

□

5.2.3. *Swarm-colony development of Proteus mirabilis.* Finally, we mention another example that fits into the abstract framework of (1.1)-(1.4). The model describes the swarming phenomenon of a bacterium called *Proteus mirabilis*. It models the evolution of mononuclear "swimmers" with density v and multi-cellular

"swarmers" with density u and reads

$$\partial_t u + \partial_a u \;=\; \mathrm{div}_x\big(D(\bar u(t,x))\,\nabla_x u\big) \;-\; m(a)\,u\;, \qquad (t,a,x)\in\mathbb{R}^+\times\mathbb{R}^+\times\Omega\;, \quad (5.31)$$

$$\partial_t v \;=\; \frac{1}{\tau}\big(1-\xi(v)\big)v + \int_0^\infty e^{a/\tau}\,m(a)\,u(t,a,x)\,\mathrm{d}a\;, \qquad (t,x)\in\mathbb{R}^+\times\Omega\;, \quad (5.32)$$

$$u(t,0,x) \;=\; \frac{1}{\tau}\,\xi\big(v(t,x)\big)\,v(t,x)\;, \qquad (t,x)\in\mathbb{R}^+\times\Omega\;, \quad (5.33)$$

$$u(0,a,x) \;=\; u^0(a,x)\;, \quad v(0,x) \;=\; v^0(x)\;, \qquad (a,x)\in\mathbb{R}^+\times\Omega\;, \quad (5.34)$$

$$\partial_\nu u(t,a,x) \;=\; 0\;, \qquad (t,a,x)\in\mathbb{R}^+\times\mathbb{R}^+\times\partial\Omega\;, \quad (5.35)$$

$$\bar u(t,x) \;=\; \int_0^\infty u(t,a,x)\,e^{a/\tau}\,\mathrm{d}a\;, \qquad (t,x)\in\mathbb{R}^+\times\Omega\;, \quad (5.36)$$

for some $\tau>0$. Let $g(a):=h(a):=e^{a/\tau}$. If ξ is sufficiently smooth, μ is bounded, and $2\alpha\in(1+n/p,2)$, then

$$B:=[u\mapsto\tau^{-1}\xi(v_u)v_u]\in C_b^{1-}\big(C([0,T],\mathbb{W}_{p,\mathcal{B}}^{2\alpha}),C^1([0,T],W_{p,\mathcal{B}}^{2\alpha})\big)$$

satisfies (A_3), where v_u is for a given u the solution to (5.32) with $v^0\in W^2_{p,\mathcal{B}}$. Moreover

$$B[u]\in C^1([0,T],L_p)\cap C([0,T],W^2_{p,\mathcal{B}})$$

if $u\in L_1([0,T],\mathbb{W}^2_{p,\mathcal{B}})$. Hence, we obtain from Theorem 2.2 and Propositions 2.5-2.7:

Proposition 5.6. *Suppose* (5.5), (5.15), *and let m be bounded. Further let $\xi\in C^3(\mathbb{R})$ and $p>n$. If $v^0\in W^2_{p,\mathcal{B}}$ and $u^0\in\mathbb{W}^2_{p,\mathcal{B}}\cap C^1(\mathbb{R}^+,L_p)\cap C(\mathbb{R}^+,W^2_{p,\mathcal{B}})$ are non-negative and satisfy $\xi(v^0)v^0=\tau u^0(0,\cdot)$, then there exists a unique non-negative solution*

$$v\in C^1(\mathbb{R}^+,W^{2\alpha}_{p,\mathcal{B}})\cap C(\mathbb{R}^+,W^2_{p,\mathcal{B}})\;,\quad u\in C(\mathbb{R}^+,\mathbb{W}^{2\alpha}_{p,\mathcal{B}})\cap L_{\infty,loc}(\mathbb{R}^+,\mathbb{W}^2_{p,\mathcal{B}})\;,\qquad \alpha\in(0,1)\;.$$

Moreover, u satisfies (2.10), (2.11) *with $E_0=L_p$.*

For details we refer to [9], in particular also for the (more realistic) case of degenerate diffusion.

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